Crystalline comparison for $A\Omega_{\mathfrak{X}}$ via $W\Omega$

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1 Introduction

Let's briefly recall the setup from the last two talks: $\mathfrak{X}$ is a smooth formal scheme over $\mathcal{O}_C$, where C is a p-adic perfectoid field. Bhatt-Morrow-Scholze constructs an object $A\Omega_R$ in $D(\mathfrak{X}, A_{\mathrm{inf}})$. This object recovers several previously known cohomology theories; one comparison we will use repeatedly is the following:

Theorem 1.1. *(Hodge-Tate comparison): Let* $\operatorname{Spf} R \subset \mathfrak{X}$ *be an affine open. Set* $\widetilde{\Omega}_R = A\Omega_R \otimes^L_{A_{\mathrm{inf}},\widetilde{\theta}} \mathcal{O}_C \in D(\mathcal{O}_C)$*. This comes with a natural map* $R \to \widetilde{\Omega}_R$*,*[1] *and its cohomology ring satisfies*

$$H^*(\widetilde{\Omega}_R) \simeq \wedge^*_R \Omega^1_{R/\mathcal{O}_C} \tag{1}$$

as a graded R*-algebra.*

Last week, Joe constructed a commutative algebra object $A\Omega^{\mathrm{sm}}_{\mathfrak{X},W}$, living in an ∞-category of presheaves defined on small affine opens of $\mathfrak{X}$ and valued in the derived ∞-category of W. He also constructed a Frobenius $\widetilde{\varphi}^{\mathrm{sm}}_{\mathfrak{X},W}$ acting on this object. Our goal is to first push this pair into the world of 1-categories, and then compare it to the de Rham-Witt complex $W\Omega^*_{\mathfrak{X}_k}$ of the special fiber of $\mathfrak{X}$.

2 Passing to 1-categories

The last talk concluded with the following proposition (10.3.10): the map

$$\widetilde{\varphi}^{\mathrm{sm}}_{\mathfrak{X},W} : A\Omega^{\mathrm{sm}}_{\mathfrak{X},W} \to \varphi_* L\eta_p A\Omega^{\mathrm{sm}}_{\mathfrak{X},W} \tag{2}$$

is an isomorphism in $\mathrm{CAlg}(\mathrm{Fun}^\infty(\mathcal{U}(\mathfrak{X})^{\mathrm{op}}_{\mathrm{sm}}, D^\infty(W)))$. In particular, the value of $(A\Omega^{\mathrm{sm}}_{\mathfrak{X},W}, \widetilde{\varphi}^{\mathrm{sm}}_{\mathfrak{X},W})$ on any small open $\operatorname{Spf} R \subset \mathfrak{X}$ is an object in the fixed points ∞-category $\widehat{D^\infty(W)}_p^{\varphi_* L\eta_p}$.

We now use the fixed-point theorems of chapter 7 to push this into the world of 1-categories.

*Notes for a talk in Berkeley's number theory seminar, on Bhatt-Lurie-Mathew's paper *Revisiting the de Rham-Witt complex*.

[1] These are $E_\infty - \mathcal{O}_C$-algebras.

Proposition 2.1. *(Example 7.6.5) The ∞-category $\widehat{D^\infty(W)}_p^{\varphi_* L\eta_p}$ is equivalent to the 1-category* $\mathrm{Mod}_W(\mathbf{DC}_{\mathrm{str}})$*,*[2] *which consists of strict Dieudonné complexes such that the underlying complex is a complex of W-modules and F is φ-semilinear.*[3]

So the pair $(A\Omega^{\mathrm{sm}}_{\mathfrak{X},W}, \widetilde{\varphi}^{\mathrm{sm}}_{\mathfrak{X},W})$ determines a presheaf $A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}$ on $\mathcal{U}(\mathfrak{X})_{\mathrm{sm}}$ valued in the 1-category $\mathrm{CAlg}(\mathrm{Mod}_W(\mathbf{DC}_{\mathrm{str}}))$. We claim that this functor is even valued in strict Dieudonné W-algebras (which we're about to define). This lets us restate the main theorem: we have an isomorphism $A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W} \simeq W\Omega^{\mathrm{sm},*}_{\mathfrak{X}_k}$ of presheaves of strict Dieudonné W-algebras on $\mathcal{U}(\mathfrak{X})_{\mathrm{sm}}$. Almost all of this talk will be spent proving this result; afterwards, we will briefly explain the completed sheafification needed to extract the crystalline comparison itself.

Definition 2.2. A *strict Dieudonné W-algebra* is a strict Dieudonné algebra A^* equipped with a morphism of Dieudonné algebras $W \to A^*$, where $W = W(k)[0]$ is a Dieudonné algebra concentrated in degree 0. We let $\mathbf{DA}_{\mathrm{strW}/}$ denote the category of these, viewed as a full subcategory of $\mathrm{CAlg}(\mathrm{Mod}_W(\mathbf{DC}))$.

Proposition 2.3. *(10.3.14) The presheaf $A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}$ takes values in $\mathbf{DC}_{\mathrm{strW}/} \subseteq \mathrm{CAlg}(\mathrm{Mod}_W(\mathbf{DC}))$.*

Proof. We must show that for every small open $U \subseteq \mathfrak{X}$, $A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}(U)$ is a strict Dieudonné W-algebra. It is strict by construction, and it is concentrated in nonnegative degrees by the coconnectivity lemma from last time.[4] So the only property remaining for us to check is the congruence $Fx \equiv x^p \pmod{VA^0}$ for $x \in A^0$. This is nontrivial, so we state it as a separate proposition. (In fact we will prove the slightly stronger fact that this congruence holds mod p.) □

Proposition 2.4. *(10.3.15) Let $U = \mathrm{Spf}\, R \subseteq \mathfrak{X}$ be a small open. Then the pth-power map on $H^0(A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}(U)/p)$ agrees with the map induced by $F : A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}(U) \to A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}(U)$.*

Proof. Idea: as is often the case when we need to actually calculate anything about $A\Omega$, we will do so by passing to a perfectoid (pro-étale) cover R_∞. We will show that the relevant cohomology group embeds Frobenius-equivariantly into $R_{\infty,k}$, and we will check that the maps agree there.

If we unwind the definitions (and the fixed point proposition from Chapter 7), the Frobenius F induces the endomorphism of $A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}(U)/p = A\Omega_R \otimes^L_{A_{\mathrm{inf}}} k$ given by $\varphi_R : A\Omega_R \to A\Omega_R$ and $\varphi : k \to k$.

Since $\mathrm{Spf}\, R$ is small, it admits an étale map to some torus $\widehat{\mathbb{G}}^d_m = \mathrm{Spf}\, \mathcal{O}_C\langle T_1^{\pm 1}, \ldots, T_d^{\pm 1}\rangle$. This torus has a perfection $\widehat{\mathbb{G}}^d_{m,\infty} = \mathrm{Spf}\, \mathcal{O}_C\langle T_1^{\pm 1/p^\infty}, \ldots, T_d^{\pm 1/p^\infty}\rangle$, and we define the perfection of R by

$$R_\infty = R \widehat{\otimes}_{\mathcal{O}_C\langle T_1^{\pm 1}, \ldots, T_d^{\pm 1}\rangle} \mathcal{O}_C\langle T_1^{\pm 1/p^\infty}, \ldots, T_d^{\pm 1/p^\infty}\rangle. \quad (3)$$

[2] Perhaps this category would be better written as $\mathrm{Mod}_{\varphi_* W}(\mathbf{DC}_{\mathrm{str}})$.

[3] This is actually proved in the generality of an arbitrary ring R equipped with an arbitrary automorphism $\sigma : R \to R$, rather than W and φ specifically.

[4] Actually, the coconnectivity lemma tells us that $H^{<0}(A\Omega^{\mathrm{sm}}_{\mathfrak{X},W} \otimes^L_W W/p^n)$ vanishes. But the generalized Cartier isomorphism tells us that mod-p^n cohomology agrees with $\mathcal{W}_n$ of the complex.

From BMS, we know that this is an integral perfectoid ring, and the map $R \to R_\infty$ has the following properties:

- $R/p \to R_\infty/p$ is faithfully flat.
- $R[1/p] \to R_\infty[1/p]$ is a pro-étale Γ-torsor, with $\Gamma = \mathbb{Z}_p(1)^{\oplus d}$.
- $A\Omega_R \simeq L\eta_\mu R\Gamma(\Gamma, A_{\inf}(R_\infty))$.[5]

The last item gives us a φ-equivariant map $\eta : A\Omega_R \to A_{\inf}(R_\infty)$.

A lemma from BMS gives us a Frobenius-equivariant isomorphism $A_{\inf}(S)\widehat{\otimes}^L_{A_{\inf}} W \simeq W(S_k)$ for every perfectoid $\mathcal{O}_C$-algebra S. Setting $S = R_\infty$ allows us to identify the image of the following map:

$$
\begin{aligned}
H^0(\eta \otimes^L_{A_{\inf}} k) : H^0(A\Omega_R \otimes^L_{A_{\inf}} k) &\to H^0(A_{\inf}(R_\infty) \otimes^L_{A_{\inf}} k) && (4)\\
&= H^0((A_{\inf}(R_\infty)\widehat{\otimes}^L_{A_{\inf}} W) \otimes^L_W k) && (5)\\
&= H^0(W(R_{\infty,k}) \otimes^L_W k) && (6)\\
&= R_{\infty,k}. && (7)
\end{aligned}
$$

This is a φ-equivariant map of k-algebras, where φ acts as desired on the left and by the p-th power map on the right. So it suffices to prove that this map is injective: if so, then φ must also act by the p-th power map on the left. We do this using the Hodge-Tate comparison: we can identify the source of the map as

$$
\begin{aligned}
H^0(A\Omega_R \otimes^L_{A_{\inf}} k) &= H^0((A\Omega_R \otimes^L_{A_{\inf},\widetilde{\theta}} \mathcal{O}_C) \otimes^L_{\mathcal{O}_C} k) && (8)\\
&= H^0(\widetilde{\Omega}_R \otimes^L_{\mathcal{O}_C} k) && (9)\\
&= H^0(\Omega^*_{R/\mathcal{O}_C} \otimes^L_{\mathcal{O}_C} k) && (10)\\
&= H^0(\Omega^*_{R_k/k}) = R_k, && (11)
\end{aligned}
$$

and the map turns out to be the obvious one. The map $R \to R_\infty$ is faithfully flat, so $R_k \to R_{\infty,k}$ is too. So it is injective, and we are done. □

3 Recognition criterion for $W\Omega$

Next we give a criterion for determining whether a given presheaf of strict Dieudonné W-algebras is isomorphic to $W\Omega^{\mathrm{sm}}_{\mathfrak{X}_k}$. Essentially, this will say that any such object satisfying the Cartier isomorphism must be $W\Omega^{\mathrm{sm}}_{\mathfrak{X}_k}$.

Let A^* be a presheaf of strict Dieudonné W-algebras on $\mathcal{U}(\mathfrak{X})_{\mathrm{sm}}$, equipped with a k-algebra map

$$\eta_A : \mathcal{O}^{\mathrm{sm}}_{\mathfrak{X}_k} \to H^0(A^*/pA^*). \tag{12}$$

[5] This appeared at the end of Koji's talk. It's the main point of working on small affines: we can calculate $A\Omega_R$ by passing to a perfectoid cover and computing group cohomology.

(Recall the generalized Cartier isomorphism: for a saturated Dieudonné complex M^*, F^r induces an isomorphism $\mathcal{W}_r(M)^* \to H^*(M^*/p^r M^*)$. So the right-hand side is $\mathcal{W}_1(A)^0$.) Then $H^*(A/pA)$ is a presheaf of cdgas over k, with differential given by the Bockstein map[6], and η_A extends naturally to a map of presheaves of cdgas

$$\widetilde{\eta}_A : \Omega^{\mathrm{sm},*}_{\mathfrak{X}_k} \to H^*(A^*/pA^*). \tag{13}$$

Proposition 3.1. *(10.4.3) With the notation above, if $\widetilde{\eta}_A$ is an isomorphism, then there is a unique isomorphism $W\Omega^{\mathrm{sm}}_{\mathfrak{X}_k} \simeq A^*$ intertwining $\eta_{W\Omega}$ with η_A.*

Proof. By the universal property of $W\Omega$ (applied on each small open), η_A lifts uniquely to a map $\Psi : W\Omega^{\mathrm{sm}}_{\mathfrak{X}_k} \to A^*$ identifying η_A with $\eta_{W\Omega}$.[7][8] This lets us construct a diagram:

$$\mathcal{O}^{\mathrm{sm}}_{\mathfrak{X}_k} \xrightarrow{\eta_{W\Omega}} H^0(W\Omega^{\mathrm{sm}}_{\mathfrak{X}_k}/p) \xrightarrow{H^0(\Psi/p)} H^0(A/pA)$$

$$\Omega^{\mathrm{sm},*}_{\mathfrak{X}_k} \underset{C^{-1}}{\xrightarrow{\sim}} H^*(W\Omega^{\mathrm{sm},*}_{\mathfrak{X}_k}/p) \xrightarrow{H^*(\Psi/p)} H^*(A/pA)$$

$$W\Omega^{\mathrm{sm},*}_{\mathfrak{X}_k} \xrightarrow{\quad\Psi\quad} A^*$$

The top row here lives in the category of presheaves of $\mathbb{F}_p$-algebras, the middle row in the category of presheaves of $\mathbb{F}_p$-cdgas, and the bottom row in the category of presheaves of strict Dieudonné algebras. The middle row is the mod-p cohomology of the bottom row, and the top row is the degree-0 part of the middle row.

By construction, the composition across the top row is η_A. By the universal property of Ω (and the fact that $H^*(\Psi/p)$ commutes with the Bockstein differentials), it follows that the composition across the middle row is $\widetilde{\eta}_A$. By assumption, this is an isomorphism, so Ψ/p is a quasi-isomorphism. But the source and target of Ψ are p-complete and p-torsion-free, so Ψ is a quasi-isomorphism. Finally, the fixed-point theorems from Chapter 7 imply that a quasi-isomorphism between strict Dieudonné complexes must be an isomorphism. So Ψ is an isomorphism, and it is unique by construction. □

4 Finishing the comparison

Our goal now is to prove the theorem stated earlier:

Theorem 4.1. *(10.4.4) There is a natural isomorphism $A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W} \simeq W\Omega^{\mathrm{sm},*}_{\mathfrak{X}_k}$ of presheaves of strict Dieudonné W-algebras on $\mathcal{U}(\mathfrak{X})_{\mathrm{sm}}$.*

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[6]The Bockstein map is the connecting homomorphism in the long exact sequence associated to the short exact sequence $0 \to A/pA \to A/p^2A \to A/pA \to 0$.

[7]The original mistakenly says $A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}$ instead of A^*.

[8]By the generalized Cartier isomorphism from my first talk, we have $F : W_1(M)^* \xrightarrow{\sim} H^0(M^*/pM^*)$ for all $M^* \in \mathbf{DC}_{\mathrm{sat}}$. Since $W\Omega^{-1} = 0$, $W_1(W\Omega^*) = W\Omega^0/VW\Omega^0$.

Proof. We will apply the recognition criterion to $A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}$. This first requires identifying $H^0(A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}/p)$ and constructing a map η_A to it. In fact we will identify all $H^i(A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}/p)$ and thereby construct $\widetilde{\eta}_A$, as the method is the same. Recall that $A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}$ was constructed by pushing

$$A\Omega^{\mathrm{sm}}_{\mathfrak{X},W} = A\Omega^{\mathrm{sm}}_{\mathfrak{X}} \widehat{\otimes}^{L}_{A_{\mathrm{inf}}} W \in \mathrm{CAlg}(\mathrm{Fun}^{\infty}(\mathcal{U}(\mathfrak{X})^{\mathrm{op}}_{\mathrm{sm}}, \widehat{D^{\infty}(W)}^{\varphi_* L\eta_p})) \tag{14}$$

into the 1-category of presheaves of strict Dieudonné algebras on $\mathcal{U}(\mathfrak{X})^{\mathrm{sm}}$. It follows that

$$H^i(A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}/p) = H^i(A\Omega^{\mathrm{sm}}_{\mathfrak{X}} \otimes^{L}_{A_{\mathrm{inf}}} k), \tag{15}$$

the "specialization of $H^i(A\Omega^{\mathrm{sm}}_{\mathfrak{X}})$ to the special fiber of A_{inf}". We can identify this specialization by passing through the Hodge-Tate specialization:

$$\begin{aligned} A\Omega^{\mathrm{sm}}_{\mathfrak{X}} \otimes^{L}_{A_{\mathrm{inf}}} k &\simeq (A\Omega^{\mathrm{sm}}_{\mathfrak{X}} \otimes^{L}_{A_{\mathrm{inf}},\widetilde{\theta}} \mathcal{O}_C) \otimes^{L}_{\mathcal{O}_C} k &(16)\\ &\simeq \widetilde{\Omega}^{\mathrm{sm}}_{\mathfrak{X}} \otimes^{L}_{\mathcal{O}_C} k, &(17)\end{aligned}$$

where $\widetilde{\Omega}$ denotes the Hodge-Tate specialization.

By the Hodge-Tate comparison, on any small open $\mathrm{Spf}(R) \subset \mathfrak{X}$, we have

$$H^*(\widetilde{\Omega}_R) = \Omega^*_{R/\mathcal{O}_C} = \wedge^*_R \Omega^1_{R/\mathcal{O}_C} \tag{18}$$

as a graded R-algebra. This is locally free and thus flat over $\mathcal{O}_C$. Thus we have

$$\begin{aligned} H^i(A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}/p)(\mathrm{Spf}\,R) &= H^i(\widetilde{\Omega}^*_{\mathrm{Spf}\,R} \otimes^{L}_{\mathcal{O}_C} k) &(19)\\ &= H^i(\widetilde{\Omega}^*_{\mathrm{Spf}\,R}) \otimes_{\mathcal{O}_C} k &(20)\\ &= \wedge^i_R \Omega^1_{R/\mathcal{O}_C} \otimes_{\mathcal{O}_C} k &(21)\\ &= \wedge^i_{R_k} \Omega^1_{R_k/k} = \Omega^*_{R_k/k} &(22)\end{aligned}$$

This gives us an isomorphism of presheaves of graded commutative k-algebras

$$\widetilde{\eta}_A : \Omega^{\mathrm{sm},*}_{\mathfrak{X}_k} \xrightarrow{\sim} H^*(A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}/p), \tag{23}$$

where the left-hand side is the presheaf $\mathrm{Spf}\,R \mapsto \Omega^*_{R_k}$. In particular, looking at degree 0, we have an isomorphism of presheaves of k-algebras

$$\eta_A : \mathcal{O}^{\mathrm{sm}}_{\mathfrak{X},W} \xrightarrow{\sim} H^0(A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}/p). \tag{24}$$

This gives us almost everything we need to run the machine of the recognition criterion. The only missing piece is that we don't know $\widetilde{\eta}_A$ is actually the cdga map induced by η_A–because we don't know that it respects the differential!

The rest of the proof will consist of checking that $\widetilde{\eta}_A$ identifies the de Rham differential with the Bockstein map. To do this, we will again use the Hodge-Tate comparison of $A\Omega_R$, and (briefly) also the proof of the de Rham comparison.

Fix a small open $\operatorname{Spf} R$, and recall that the crystalline specialization map $A_{\inf} \to W$ sends $\widetilde{\xi}$ to p. This allows us to compare the Bockstein constructions for $\widetilde{\xi}$ and p with the following commutative diagram in $D(A_{\inf})$:

$$\begin{array}{ccccc}
A\Omega_R/\widetilde{\xi} \simeq A\Omega_R \otimes^L_{A_{\inf},\widetilde{\theta}} \mathcal{O}_C & \xrightarrow{\widetilde{\xi}} & A\Omega_R/\widetilde{\xi}^2 & \longrightarrow & A\Omega_R/\widetilde{\xi} \simeq A\Omega_R \otimes^L_{A_{\inf},\widetilde{\theta}} \mathcal{O}_C \\
\downarrow & & \downarrow & & \downarrow \\
A\Omega_R/\widetilde{\xi} \otimes^L_{A_{\inf}} W & \xrightarrow{\widetilde{\xi}} & A\Omega_R/\widetilde{\xi}^2 \otimes^L_{A_{\inf}} W & \longrightarrow & A\Omega_R/\widetilde{\xi} \otimes^L_{A_{\inf}} W \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
(A\Omega_R \widehat{\otimes}^L_{A_{\inf}} W) \otimes^L_{\mathbb{Z}_p} \mathbb{F}_p & \xrightarrow{p} & (A\Omega_R \widehat{\otimes}^L_{A_{\inf}} W) \otimes^L_{\mathbb{Z}_p} \mathbb{Z}/p^2 & \longrightarrow & (A\Omega_R \widehat{\otimes}^L_{A_{\inf}} W) \otimes^L_{\mathbb{Z}_p} \mathbb{F}_p\,.
\end{array}$$

The rows of this diagram are exact triangles, and the second row is just the first one $\otimes^L_{A_{\inf}} W$. So we get a commutative diagram of connecting maps from the first and last rows:

$$\begin{array}{ccc}
H^i(A\Omega_R \otimes^L_{A_{\inf},\widetilde{\theta}} \mathcal{O}_C) & \xrightarrow{\beta_{\widetilde{\xi}}} & H^{i+1}(A\Omega_R \otimes^L_{A_{\inf},\widetilde{\theta}} \mathcal{O}_C) \\
\downarrow & & \downarrow \\
H^i((A\Omega_R \widehat{\otimes}^L_{A_{\inf}} W) \otimes^L_{\mathbb{Z}_p} \mathbb{F}_p) & \xrightarrow{\beta_p} & H^{i+1}((A\Omega_R \widehat{\otimes}^L_{A_{\inf}} W) \otimes^L_{\mathbb{Z}_p} \mathbb{F}_p).
\end{array}$$

The Hodge-Tate comparison identifies this with:

$$\begin{array}{ccc}
\Omega^i_{R/\mathcal{O}_C} & \xrightarrow{\beta_{\widetilde{\xi}}} & \Omega^{i+1}_{R/\mathcal{O}_C} \\
\downarrow & & \downarrow \\
\Omega^i_{R_k} & \xrightarrow{\beta_p} & \Omega^{i+1}_{R_k}
\end{array}$$

The bottom map is the Bockstein in question, so it suffices to prove that the top map coincides with the de Rham differential. This is done in (Bhatt's companion to) BMS. Idea: reduce to studying $\widehat{\mathbb{G}}_m^d$ by our étale map, then reduce to $d = 1$ by the Künneth formula for differentials. Then do a group cohomology computation, where the non-integral degree parts go away and the integral degree parts can be calculated explicitly with a Koszul complex. □

Theorem 4.2. *(Main comparison, first version, 10.2.1): There is a natural identification* $A\Omega_{\mathfrak{X}} \widehat{\otimes}_{A_{\inf}} W \simeq W\Omega^*_{\mathfrak{X}_k}$ *of commutative algebras in* $D(\mathfrak{X}, W)$ *that carries* $\varphi_{\mathfrak{X},W}$ *to* $\varphi_{\mathfrak{X}_k}$.

Proof. We now have an isomorphism $\Psi : W\Omega^{\mathrm{sm},*}_{\mathfrak{X}_k} \to A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}$ of presheaves of strict Dieudonné W-algebras on $\mathcal{U}(\mathfrak{X})^{\mathrm{sm}}$. Passing to the derived category gives a φ-equivariant isomorphism $\Theta : W\Omega^{\mathrm{sm},*}_{\mathfrak{X}_k} \to A\Omega^{\mathrm{sm}}_{\mathfrak{X},W}$ of commutative algebra objects in $\operatorname{Fun}(\mathcal{U}(\mathfrak{X})^{\mathrm{op}}_{\mathrm{sm}}, \widehat{D}(W))$. The objects we are interested in are the completed sheafifications of the two sides; i.e. their images under the functor

$$\widehat{G}_W(-) = \lim_{\leftarrow n} \left((- \otimes^L_W W/p^n)^{\mathrm{sh}} \right). \tag{25}$$

So applying $\widehat{G}_W$ to Θ proves the theorem. □